MATH2050C Selected Solution to Assignment 8

Section 4.1

(9d). We use ε-δ definition. Consider

$$\left|\frac{x^2-x+1}{x+1}-\frac{1}{2}\right| = \left|\frac{2x^2-3x+1}{2(x+1)}\right| = \left|\frac{2x-1}{x+1}\right||x-1|.$$

We make a first choice $\delta_1 = 1/2$. Then for $|x-1| < 1/2$, that is, $1/2 < x < 3/2$. Then $|2x-1|/|x+1| \le 4/3$. Therefore, for $\delta = \min\{\delta_1, 3\varepsilon/4\}$, we have

$$\left|\frac{x^2-x+1}{(x+1)-1/2}\right| < \frac{4}{3}|x-1| < \varepsilon \ ,$$

for $x, 0 < |x-1| < \delta$.

Or, we could use Sequential Criterion. Let $\lim_{n\to\infty} x_n = 1$. By Limit Theorem $\lim_{n\to\infty}(x_n^2 - x_n + 1) = 1$ and $\lim_{n\to\infty}(x_n+1) = 2$. Therefore,

$$\lim_{n\to\infty} \frac{x_n^2-x_n+1}{x_n+1} = \frac{\lim_{n\to\infty}(x_n^2-x_n+1)}{\lim_{x\to\infty}(x_n+1)} = \frac{1}{2} \ .$$

(12d). We claim that $\lim_{x\to 0}\sin(1/x^2)$ does not exist. Take the sequence $x_n = \sqrt{1/(2n\pi)}$ and $y_n = \sqrt{1/(2n\pi+\pi/2)}, n \ge 1$. Both sequences tend to 0 as $n \to \infty$. As $\lim_{n\to\infty}\sin(1/x_n^2) = 0$ and $\lim_{n\to\infty}\sin(1/y_n^2) = 1$, they have different limit. We conclude that the limit of $\sin(1/x^2)$ as $x \to 0$ does not exist.

(15). (a) We want to show $\lim_{x\to 0} f(x) = 0$ where f is the function that is equal to x at rational x and 0 at irrational x. The desired conclusion follows from the observation $|f(x)| \le |x|$ and $\lim_{x\to 0}|x| = 0$ and the Squeeze Theorem.

(b) f has no limit at $x = c \neq 0$. Let $x_n \to c$ be a sequence of rational numbers. Clearly, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = c$. But, take $y_n \to c$ be a sequence of irrational numbers, then $f(y_n) = 0$, so $\lim_{n\to\infty} f(y_n) = 0$. From Sequential Criterion we draw the desired conclusion.

Section 4.2 no. 1bc, 11 cd, 12.

(1b). Since the limit is taken among positive x only, this should be viewed as a right limit (see below). By Limit Theorem,

$$\lim_{x\to 1^+}\frac{x^2+2}{x^2-2} = \frac{\lim_{x\to 1^+}(x^2+2)}{\lim_{x\to 1^+}(x^2-2)} = \frac{3}{-1} = -3.$$

(11c). Let $x_n = 1/(2n\pi) \to 0$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \operatorname{sgn}\sin 1/x_n = \lim_{n\to\infty} \operatorname{sgn} 0 = 0 \ .$$

On the other hand, let $y_n = 1/(2n\pi + \pi/2) \to 0$ as $n \to \infty$. We have

$$\lim_{n\to\infty} \operatorname{sgn}\sin 1/y_n = \lim_{n\to\infty} \operatorname{sgn}\ 1 = 1.$$

We conclude from Sequential Criterion that the limit does not exist.

(11d). Using the inequality

$$|\sqrt{x}\sin 1/x^2| \le \sqrt{x} \ ,$$

and the fact that $\lim_{x\to 0^+} \sqrt{x} = 0$, we conclude from Squeeze Theorem that

$$\lim_{x\to 0^+} \sqrt{x} \sin \frac{1}{x^2} = 0 \ .$$

Supplementary Problems

1. Suppose that for a polynomial p and $c \in \mathbb{R}$, prove by the Limit Theorem (see next page) that $\lim_{x\to c} p(x) = p(c)$.

   Solution. Using $\lim_{x\to c} x = c$ for all c, the product rule tells us that $\lim_{x\to c} x^k = (\lim_{x\to c} x)(\lim_{x\to c} x)\cdots(\lim_{x\to c} x) = c^k$ (k-times product). Hence

$$\begin{aligned} \lim_{x\to c} p(x) &= \lim_{x\to c}(a_0 + a_1 x + \cdots + a_n x^n) \\ &= a_0 \lim_{x\to c} 1 + a_1 \lim_{x\to c} x + \cdots + a_n \lim_{x\to c} x^n \\ &= a_0 + a_1 c + \cdots + a_n c^n \\ &= p(c) \ . \end{aligned}$$

2. Let f be defined on (a, b) possibly except $x_0 \in (a, b)$. Show that $\lim_{x\to x_0} |f(x)| = |L|$ whenever $\lim_{x\to x_0} f(x) = L$.

   Solution. It follows immediately from the triangle inequality $||f(x)| - |L|| \le |f(x) - L|$.

3. Let f be defined on (a, b) possibly except $x_0 \in (a, b)$. Suppose that $\lim_{x\to x_0} f(x) = L$ for some L. Show that $\lim_{x\to x_0} \sqrt{f(x)} = \sqrt{L}$ provided $f \ge 0$ on (a, b). Suggestion: Consider $L > 0$ and $L = 0$ separately.

   Solution. First, assume $L > 0$. Given $\varepsilon = L/2 > 0$, there is some δ_1 such that $|f(x)-L| \le L/2$ for $0 < |x - x_0| < \delta_1$. In particular, it implies that $f(x) \ge L/2$ for $0 < |x - x_0| < \delta_1$. Now,

$$|\sqrt{f(x)} - L^{1/2}| = \frac{|f(x) - L|}{\sqrt{f(x)} + L^{1/2}} \le \frac{1}{(L/2)^{1/2} + L^{1/2}} \times |f(x) - L| \ ,$$

   for $0 < |x - x_0| < \delta_1$. For $\varepsilon > 0$, there is δ_2 such that $|f(x) - L| < \varepsilon \times [(L/2)^{1/2} + L^{1/2}]$ for $x, 0 < |x - x_0| < \delta_2$. If we take $\delta = \min\{\delta_1, \delta_2\}$, then

$$|\sqrt{f(x)} - L^{1/2}| < \frac{1}{(L/2)^{1/2} + L^{1/2}} \times |f(x) - L| < \varepsilon \ , \quad \forall x, 0 < |x - x_0| < \delta,$$

   done.

   Next, $L = 0$. Given $\varepsilon > 0$, there is some δ such that $|f(x)| < \varepsilon^2$ for all $x, 0 < |x - x_0| < \delta$. It follows that $|\sqrt{f(x)} - 0| = \sqrt{f(x)} < \varepsilon$ for $x, 0 < |x - x_0| < \delta$, done.

4. Let f be function defined on (a, b) except possibly at $x_0 \in (a, b)$. It is has a **right hand limit** at x_0 if there exists some L such that for all $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in (x_0, x_0 + \delta) \cap (a, b)$. Denote it by $L = \lim_{x\to x_0^+} f(x)$. Similarly we define the **left hand limit** of f at x_0 and denote it by $\lim_{x\to x_0^-} f(x)$. Show that $\lim_{x\to x_0} f(x)$ exists if and only if both one-sided limits exist and are equal.

   Solution. $\Rightarrow$. When $\lim_{x\to x_0} f(x) = L$, for $\varepsilon > 0$, there is some δ such that $|f(x) - L| < \varepsilon$ for $0 < |x - x_0| < \delta, x \in (a, b)$. Certainly it means $|f(x) - L| < \varepsilon$ for $x \in (x_0, x_0 + \delta), x \in$

(x_0, b), and $x \in (x_0 - \delta, x_0), x \in (a, x_0)$. In other words, both one-sided limits exist and equal.

$\Rightarrow$. Let $L = \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x)$. For $\varepsilon > 0$, there exists δ_1 such that $|f(x) - L| < \varepsilon$ for $x \in (x_0, x_0 + \delta_1), x \in (x_0, b)$. On the other hand, there exists δ_2 such that $|f(x) - L| < \varepsilon$ for $x \in (x_0 - \delta_2, x_0), x \in (a, x_0)$. Therefore, by taking $\delta = \min\{\delta_1, \delta_2\}$, $|f(x) - L| < \varepsilon$ for all $x \in (a, b), 0 < |x - x_0| < \delta$, that is, $\lim_{x \to x_0} f(x) = L$.